Graph Topology and Discrete Morse Theory

Andrew Rosevear, Andrew Tawfeek
Advised by Prof. Ivan Contreras

Amherst College

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Outline Part I

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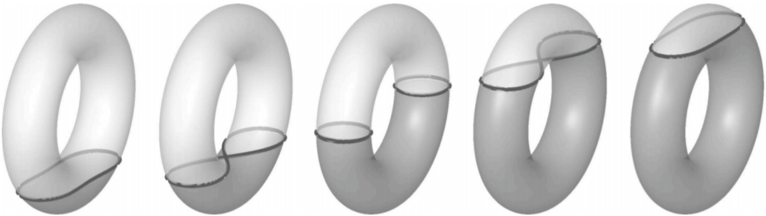
Smooth Morse Theory

Building Intuition



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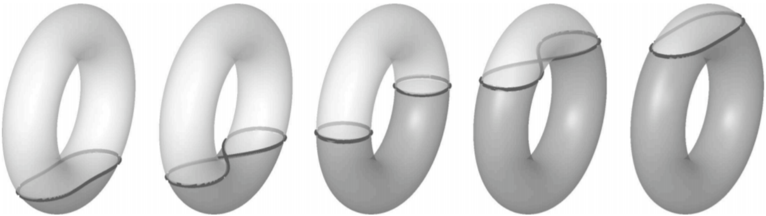

Note that an important result in smooth Morse theory is that given a critical point, we can choose the correct local coordinates so the function takes the form of a parabaloid opening upwards/downwards or a saddle point.

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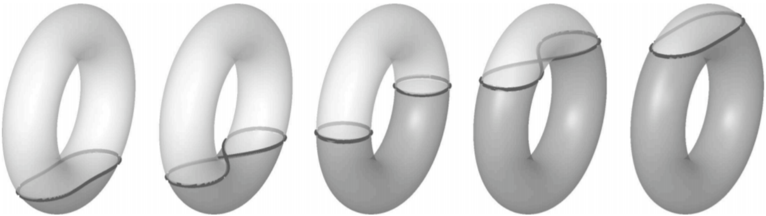

Note that an important result in smooth Morse theory is that given a critical point, we can choose the correct local coordinates so the function takes the form of a parabaloid opening upwards/downwards or a saddle point.

Lastly, it turns out that there is an important correspondence between Morse functions f and gradient-like vector fields for f.

Shifting View

Discrete Morse theory was developed by Robin Forman around 2002, in his published work *A Users Guide to Discrete Morse Theory*.

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We will only focus on the definition with 0-cells (vertices) and 1-cells (edges), i.e. graphs.

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Focusing on Graphs

Definitions

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Definition

Let $\Gamma = (V, E)$ be a graph. A *discrete Morse function* is a function $f : \Gamma_c = V \cup E \to \mathbb{R}$ such that for every $\sigma \in \Gamma_c$,

$$|\{\tau \in \Gamma_c \mid \sigma < \tau \text{ and } f(\sigma) \geq f(\tau)\}| \leq 1; \tag{1}$$

$$|\{\tau \in \Gamma_c \mid \sigma > \tau \text{ and } f(\sigma) \leq f(\tau)\}| \leq 1. \tag{2}$$

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We say a cell (vertex or edge) is *critical* if both sets (1) and (2) above are empty. We let $c_0(f)$ denote the number of critical vertices, and $c_1(f)$ number of critical edges.

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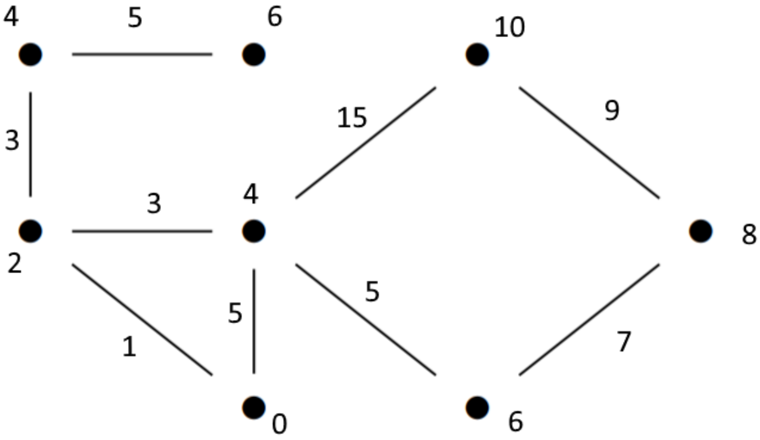
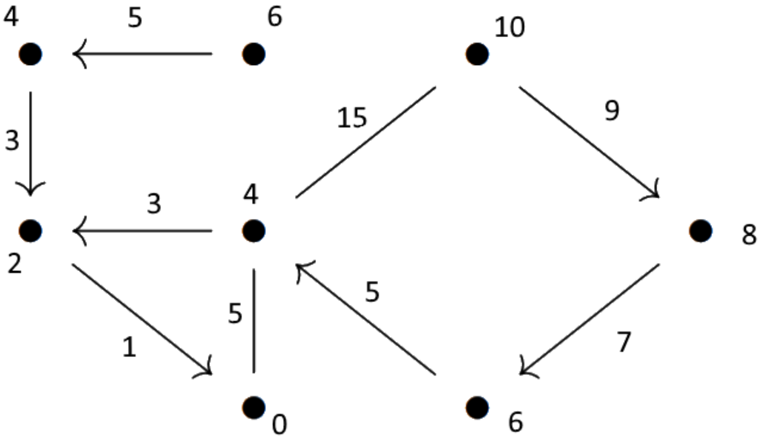


Results

Gradient Flow



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Characterizing Critical Cells

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Theorem (A.T.)

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In fact, we can do even better. We can fully characterize discrete Morse functions with gradient flows.

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Characterizing Gradient Flow

Theorem (A.T)

Let Γ_o be a directed graph and Γ be it's underlying undirected graph. Then $\Gamma_o = \Gamma_f$ for some discrete Morse function f on Γ if and only if

1. *no two edges share a tail, and*
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If we denote by Morse(Γ) be the set of all discrete Morse functions that can be defined on Γ, then

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This turns out to be equivalent to Forman's equivalence of discrete Morse functions on a graph, i.e. f is equivalent to g if and only if for every vertex and edge of Γ,

$$f(v) < f(e) \iff g(v) < g(e).$$

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Additionally, there have been numerous technical results concerning the equivalence classes that are still being studied in more depth.

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- By observing the barycentric subdivision of simiplicial complexes and regular CW complexes, attempt to frame general discrete Morse theory in terms of these equivalence classes.
- Develop an analogue of discrete Morse theory for hypergraphs.

Bibliography

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Outline

Graph Chain Complexes

Definition (Chain Complex of a Graph)

The **chain complex of a directed graph** $\Gamma = (V, E)$ is a sequence of vector spaces paired with linear maps

$$0 \to \mathbb{C}^{|E|} \xrightarrow{\partial_1} \mathbb{C}^{|V|} \to 0,$$

where ∂_1 is the **boundary operator** given by the $|V| \times |E|$ **incidence matrix** I whose entries are

$$I_{ij} = \begin{cases} 1 & \text{edge } e_j \text{ enters vertex } v_i \\ -1 & \text{edge } e_j \text{ leaves vertex } v_i \\ 0 & \text{otherwise} \end{cases}.$$

The Graph Laplacian

Definition (Graph Laplacian)

The **even and odd graph Laplacians** Δ^+ and Δ^- of an oriented graph Γ are given by

$$\Delta^+ := II^* : \mathbb{C}^{|V|} \to \mathbb{C}^{|V|}$$

$$\Delta^- := I^*I : \mathbb{C}^{|E|} \to \mathbb{C}^{|E|}.$$

Both matrices are positive semidefinite and symmetric (and therefore diagonalizable).

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Both matrices are positive semidefinite and symmetric (and therefore diagonalizable).

Lemma

Δ^+ *is invariant under orientation. However,* Δ^- *is not.*

Homology and Betti Numbers

Definition (Homology Groups)

The **homology groups** of a graph Γ are given by

$$H_1(\Gamma) = \ker(I) = \ker(\Delta^-)$$

and

$$H_0(\Gamma) = \ker(I^*) = \ker(\Delta^+).$$

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Theorem (Contreras-Xu)

Let b_1 and b_0 be the Betti numbers of Γ. Then

$$\dim(H_1(\Gamma)) = b_1$$

$$\dim(H_0(\Gamma)) = b_0.$$

Cochain Complexes and the Graph Differential

The **graph cochain complex** is simply the graph chain complex but with the arrows reversed, and I replaced with I^*:

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Cochain Complexes and the Graph Differential

The **graph cochain complex** is simply the graph chain complex but with the arrows reversed, and I replaced with I^*:

$$0 \to \mathbb{C}^{|V|} \xrightarrow{I^*} \mathbb{C}^{|E|} \to 0.$$

By analogy with the differential operator on the de Rham complex, we may view I^* as a kind of graph differential operator.
In vector calculus terminology, I^* serves as the graph gradient.

Integration on Graphs

Definition (Vertex Integral)

Let f be a discrete function on the vertices of Γ. The **vertex integral of f over Γ** is given by

$$\int_{\partial\Gamma}^{\bullet} f = \sum_{v_i \in V} f(v_i) d_n(v_i),$$

where $d_n(v_i)$ is the number of incoming minus outgoing edges.

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Definition (Edge Integral)

Let F be a discrete function on the edges of Γ. The **edge integral of F over Γ** is given by

$$\int_{\Gamma}^{-} F = \sum_{e_i \in E} F(e_i).$$

Main Results

Theorem (A.R)

(Stokes' Theorem for Graphs) *Let f be a vertex function on an oriented graph Γ. Then*

$$\int_{\partial\Gamma}^{\bullet} f = \int_{\Gamma}^{-} I^* f.$$

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Theorem (A.R)

(Graph Hodge Decomposition) *Let Γ be an oriented graph and $H_0(\Gamma) = \ker(\Delta^+) = \ker(I^*)$ and $H_1(\Gamma) = \ker(\Delta^-) = \ker(I)$ its nth homology group. Then*

$$\mathbb{C}^{|V|} = H_0(\Gamma) \oplus \mathrm{Im}(I)$$

$$\mathbb{C}^{|E|} = H_1(\Gamma) \oplus \mathrm{Im}(I^*).$$

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3. A more coherent theory of graph calculus can help answer questions about graph differential equations, such as the graph Schrodinger equation

$$\frac{\partial \varphi}{\partial t} = i\Delta\varphi.$$